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ON THE MINIMIZATION OF RESIDUAL THERMAL STRESSES IN VISCOPLASTIC MATERIALS

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Abstract—This article demonstrates the feasibility of the existence of optimal cooling of materials which exhibit viscoplastic response. Accordingly, cooling those materials along specific temperature–time paths would yield minimal residual thermal stresses upon termination of cool-down.

1. INTRODUCTION

Residual thermal stresses develop when geometrically constrained materials are subjected to thermal excursions. These residual stresses are of particular interest in the case of composite materials, where geometric constraints are encountered at several levels; on the "micro"-scale there exists a mismatch between the thermo-mechanical properties of filler and matrix materials, on the "mini" scale one notices the mutual constraints that plies with fibers directed at distinct orientations impose upon each other, while structural configurations induce constraints on the "macro" scale.

In both polymeric and metal matrix composites the more pliable matrix materials exhibit non-linear stress–strain responses in the high stress and elevated temperature ranges. In several circumstances these responses were modeled by means of viscoelasticity or viscoplasticity theories.

In the case of viscoelastic response, as may apply to polymeric resins, it has been shown that it is possible to minimize residual thermal stresses by following specific optimal temperature–time cooling paths (Gurtin and Murphy, 1980; Weitsman, 1980; Weitsman and Harper, 1982; Harper, 1985; Lee and Weitsman, 1992).

The present work demonstrates that it is also possible to obtain an optimal temperature–time cooling path in the case of rate-dependent plastic response. Although the existence of optimal cooling paths is demonstrated herein for an idealized, specific formulation of viscoplastic response, there is good reason to expect that analogous optimal paths would exist in a wider range of response representations because the model employed herein possesses many of the essential features of rate-dependent plastic behavior.

2. FORMULATION

To demonstrate the feasibility of optimal cooling in rate-dependent plastic materials we consider, as a point of departure, the simplified circumstance of a one-dimensional, plastic-like, rate independent, stress–strain behavior expressed by

$$\sigma = S \tanh\left(\frac{E}{S}\varepsilon\right). \tag{1}$$

The above relation reduces to linear elasticity for infinitesimally small strains ε and corresponds to a yield stress σ of magnitude S.

When the foregoing material is subjected to temperature variations in the presence of geometric constraints, it will develop residual thermal stresses according to eqn (1) where ε accounts for the stress induced portion of the total strain $e(e = \varepsilon + \varepsilon_{\text{thermal}})$.

Rate and temperature dependence are introduced into the relation given in eqn (1) by assuming that both S and E depend on temperature T and strain-rate $\dot{\varepsilon} = \mathrm{d}\varepsilon/\mathrm{d}t$.

Converting eqn (1) to an incremental relation, we have

$$\mathrm{d}\sigma = \frac{E(T,\dot{\varepsilon})}{\cosh^2\left[\dfrac{E(T,\dot{\varepsilon})}{S(T,\dot{\varepsilon})}\varepsilon\right]}\mathrm{d}\varepsilon. \tag{2}$$

The presence of rate dependence in the incremental stress–strain response, expressed in eqn (2), is the key factor which enables the attainment of cool-down optimization.

The optimization problem at hand is formulated as follows:

Given an initial, elevated, temperature T_I, at which the material is stress free, a final temperature $T_F(T_I > T_F)$ to be reached at a prescribed cooling time t_F, namely $T(t_F) = T_F$, and a material whose response is expressed by eqn (2), find the optimal temperature–time path $T = T_\Omega(t)$ which yields a minimal residual stress $\sigma(t_F)$.

In order to proceed it is necessary to further characterize the functions $E(T,\dot{\varepsilon})$ and $S(T,\dot{\varepsilon})$. Based upon physical reasoning we consider E and S to increase with $|\dot{\varepsilon}|$ and decrease with T, and select

$$E = E_0 \exp[-\lambda_1(T-T_F)+\lambda_2|\dot{\varepsilon}|], \tag{3a}$$

$$S = S_0 \exp[-\mu_1(T-T_F)+\mu_2|\dot{\varepsilon}|]. \tag{3b}$$

In eqns (3), λ_1, λ_2, μ_1 and μ_2 are positive constants and E_0, S_0 are the values of E and S, respectively, at the final temperature T_F and during extremely slow processes.

Consequently, in view of eqns (2) and (3), we obtain the following expression for $\sigma(t_F)$:

$$\sigma(t_F) = \int_0^{t_F} \frac{E_0 \exp[-\lambda_1(T-T_F)+\lambda_2|\dot{\varepsilon}|]\dot{\varepsilon}\,\mathrm{d}t}{\cosh^2\left\{\dfrac{E_0}{S_0}\exp[-(\lambda_1-\mu_1)(T-T_F)+(\lambda_2-\mu_2)|\dot{\varepsilon}|]\varepsilon\right\}}. \tag{4}$$

Equation (4) expresses $\sigma(t_F)$ as a functional of $T(t)$, $\varepsilon(t)$ and $\dot{\varepsilon}(t)$.

When both geometric constraint and the coefficient of thermal expansion α depend at most on T, we have $e = e(T)$, $\varepsilon_{\text{thermal}} = \alpha(T)(T_I - T) = \varepsilon_{\text{th}}(T)$ and therefore also $\varepsilon = e(T) - \varepsilon_{\text{th}}(T) = \varepsilon(T)$.

In this circumstance, which corresponds to most realistic cases, expression (4) reduces to a functional in $T(t)$ and $\dot{T}(t)$.

In the computational examples we shall consider, for simplicity, a rigid constraint, namely $e(T) = 0$, as well as a constant coefficient of thermal expansion. Consequently we shall have $\varepsilon = -\alpha(T_I - T)$.

The latter simplifications reduce the computational tasks discussed in the sequel but do not impose any essential limitation on the feasibility of optimal cooling.

3. OPTIMIZATION

Consider $\varepsilon = -\alpha(T - T_I)$ in expression (4) with prescribed values of temperatures $T(t=0) = T_I$ and $T(t = t_F) = T_F$.

In this case, expression (4) reduces to

$$\sigma(t_F) = \int_0^{t_f} F(T, \dot{T})\,\mathrm{d}t, \tag{5}$$

where $F(T, \dot{T})$ is given by the integrand in eqn (4), with $\varepsilon = -\alpha(T - T_I)$ and $\dot{\varepsilon} = -[\alpha'(T)(T - T_I) + \alpha]\dot{T}$.

A stationary value of $\sigma(t_F)$ corresponds to the optimal temperature–time path $T(t) = T_\Omega(t)$ which satisfies the well-known Euler's equation

$$\frac{\partial F}{\partial T} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial F}{\partial \dot{T}}\right) = 0. \tag{6}$$

For a minimum of $\sigma(t_F)$ it is necessary to have

$$\left.\frac{\partial^2 F}{\partial \dot{T}^2}\right|_{T = T_\Omega(t)} > 0. \tag{7}$$

In the present case, the resulting Euler's equation is a non-linear, second order, ordinary differential equation for $T(t)$. With prescribed end conditions $T(0) = T_I$, $T(t_f) = T_F$ this equation can be solved by numerical iteration.

Alternatively, upon noting that $(\mathrm{d}/\mathrm{d}t) = \dot{T}(\mathrm{d}/\mathrm{d}t)$, it is possible to obtain the first integral of eqn (6), which reads

$$F(T, \dot{T}) - \dot{T}\frac{\partial F(T, \dot{T})}{\partial \dot{T}} = C, \tag{8}$$

where C is an arbitrary constant.

More specifically, in the present case, eqn (8) takes the form†

$$\frac{\alpha E(T, \dot{T})\dot{T}^2\left\{-\lambda_2 + 2\tanh\left[\dfrac{E(T, \dot{T})}{S(T, \dot{T})}\varepsilon\right](\lambda_2 - \mu_2)\dfrac{E(T, \dot{T})}{S(T, \dot{T})}\varepsilon\right\}}{\cosh^2\left[\dfrac{E(T, \dot{T})}{S(T, \dot{T})}\varepsilon\right]} = C, \tag{9}$$

where $E(T, \dot{T})$ and $S(T, \dot{T})$ are given in eqns (3) and, upon assuming rigid geometric constrains, $\varepsilon = -\alpha(T - T_I)$.

Note that the time t is not explicitly present in eqns (8) and (9). At this stage, select a guess value of $\dot{T}(0)$, say $\dot{T}_{og}$. Substitution of this value of $\dot{T}_{og}$, together with $T(0) = T_I$, into eqn (9) determines a value of the constant C which, obviously, remains fixed for all subsequent temperatures.

Next, discretize the temperature interval $T_F < T < T_I$ into, say, n equal sub-intervals and consider intermediate values $T_i = T_I - (i/n)(T_I - T_F)$ $(i = 0, 1, \ldots, n)$.

At each T_i, the expression $F(T_i, \dot{T}_i) - \dot{T}_i(\partial F/\partial \dot{T}_i)(T_i, \dot{T}_i) = C$ yields a non-linear, transcendental, relation for $\dot{T}_i$, which is solved by standard computational routines.‡

The n values of $\dot{T}_i$ $(i = 0, 1, \ldots, n-1)$ determine a corresponding duration t_F of the cooling time, given by

†Note that ε, not $|\varepsilon|$, appears in eqn (9), because the temperature along the optimal cooling path drops monotonically with time.

‡The explicit expression for $\partial F/\partial \dot{T}$ is obtainable in a straightforward manner, and is omitted here for the sake of brevity.

$$t_F(\dot{T}_{og}) = \int_{T_I}^{T_F} \frac{dT}{\dot{T}(T)} \cong -\frac{T_I - T_F}{n} \sum_{i=0}^{n-1} \frac{1}{\dot{T}_i}. \tag{10}$$

The guess value of $\dot{T}(0)$ is subsequently adjusted until $t_F(\dot{T}_0)$ agrees with the prescribed value of the cooling time t_F to within a desired accuracy.

4. COMPUTATIONAL RESULTS

To illustrate the feasibility of optimal cooling which minimizes residual thermal stresses we considered the following numerical values for the parameters listed in eqns (3):

$E_0 = 100$ GPa, $S_0 = 1$ GPa, $\alpha = 10^{-5}\,°C^{-1}$, $\lambda_1 = \mu_1 = 0.092\,°C^{-1}$, $\lambda_2 = \mu_2 = 10^3$ time units $°C^{-1}$, and with $t_F = 1, 2$ and 3 time units.

In addition, we took $T_I = 120°C$ and $T_F = 20°C$ and considered cooldown against rigid geometric constraints, namely $\varepsilon = -\alpha(T - T_I)$.

The optimal cooling paths $T_\Omega(t)$ for $t_F = 1$, 2 and 3 are shown versus time in Fig. 1, while the gradual build-up in the residual thermal stresses versus time is exhibited in Fig. 2 for three distinct values of cooling times $t_F = 1, 2$ and 3. These are plotted in solid lines.

For comparison purposes the residual stresses under constant cooling rates are shown in dashed lines in Fig. 2.

Note that in all three cases optimal cooling yielded lower residual thermal stresses than constant rate cooling. In addition, as may be expected, extended cooling times t_F yielded smaller residual stresses.

For the specific forms [eqns (1) and (3)] and parametric values considered herein, it was possible to establish numerically the validity of eqn (7), thereby the existence of minimal values of residual stresses.

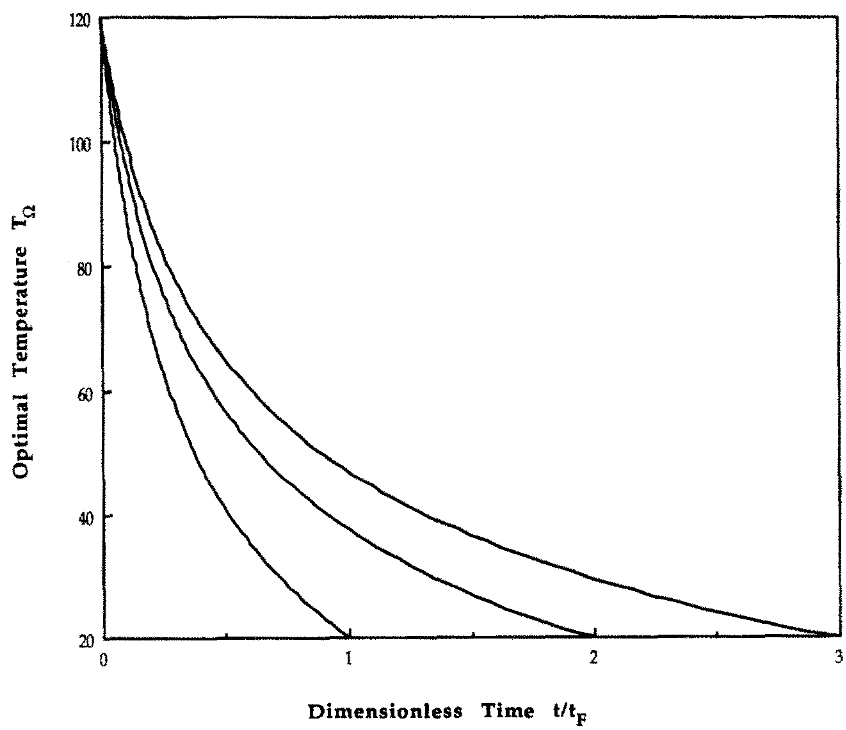

Fig. 1. Optimal cooling temperature $T_\Omega(t)$ vs time for the parametric values listed in the text, with $t_F = 1, 2$ and 3 (units of time).

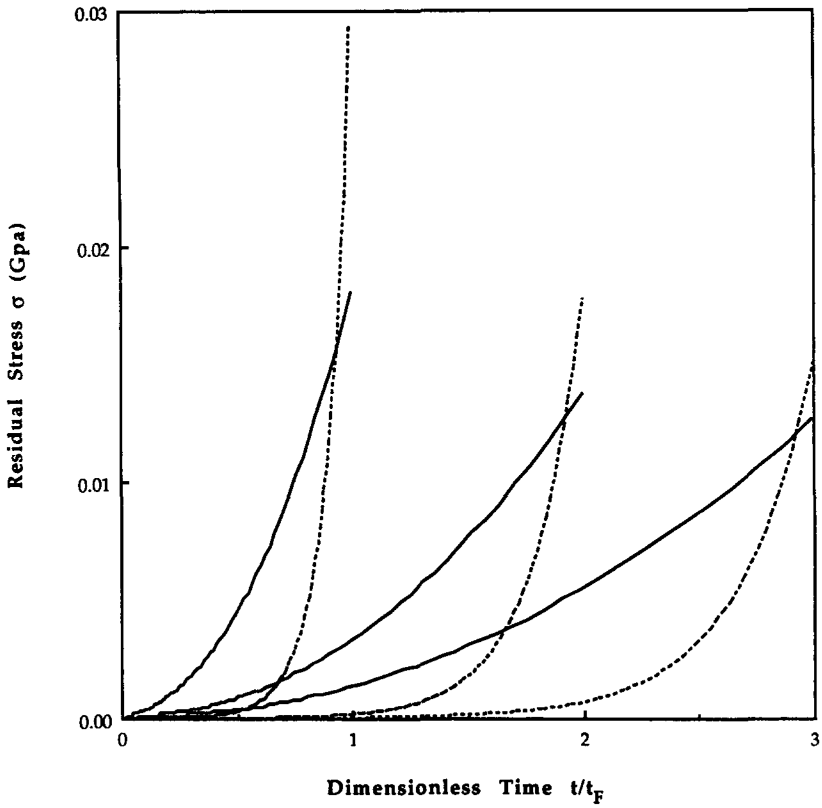

Fig. 2. The build-up of residual thermal stresses which correspond to optimal cooling (solid lines) and constant rate cooling (dashed lines), for the parametric values listed in the text and with $t_F = 1$, 2 and 3 (units of time).

5. CONCLUSIONS

The results of this paper demonstrate that a minimization of residual thermal stresses is feasible in viscoplastic materials. It was shown that these stresses, which arise during cooling from an elevated, manufacturing temperature down to a lower, operational temperature can be minimized by following a specific temperature–time cool-down path.

The existence of such an optimal cooling path was established previously for the cases of linear and non-linear viscoelastic responses. The present work suggests that similar optima exist for materials which exhibit any type of rate-dependent response. However, the validity of the above suggestion has been demonstrated herein only for a specific circumstance and remains to be verified in other cases.

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